THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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LONDON:

GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN, AND BOMBAY.

Vol. IV. January, 1908.—Part II. No. 69.

THE FIRST LOCAL BRANCH OF THE MATHEMATICAL ASSOCIATION.

We are glad to report that the first Local Branch has been started at Bangor, and that its first President is Prof. G. H. Bryan, F.R.S., the President of the Association for 1907. The Branch consists of Members and Associates. Members must be members of the Mathematical Association, save in the case of teachers from the same school or members of the same family, when it is not necessary that more than one shall be a member of the parent body. The qualification for Associates is the payment of a small subscription. Apart from the interest attaching to the foundation of the first local branch, there is reason for congratulation in the fact that the Branch has been formed for "the discussion of matters relating to the teaching of mathematics in schools, etc., *of all grades.*" Here is an important link in the long chain that has yet to be forged before the teaching profession in this country becomes one organic whole.

THE INTRODUCTION OF IRRATIONAL NUMBERS.

The subject of the definition of irrational numbers is so intimately connected with questions about the existence of a limit of a sequence, which has recently formed the subject of much discussion in this *Gazette*,* that a fuller consideration of the subject, and, in particular, of how irrational numbers are to be introduced into a course introductory to 'higher' mathematics, may not be undesirable.

1.

When we have introduced the positive and negative integer numbers and rational numbers, whether in a purely arith-

* October, 1905, pp. 236-237; May, 1906, p. 327; July, 1906, pp. 333-335, 349-350 October, 1906, p. 380 (all of vol. iii.).

metical manner, as relations (or operations),* or as certain lengths † on a straight line, we come to the consideration of infinite processes. We define a *limit* of a *convergent* ‡ sequence, and find that, while some sequences have (rational) limits, some, although convergent, or, what is the same thing, such that $|s_n|$ is always (however great n is) less than a fixed rational number (and therefore an infinity of them), have not. On the other hand, if we compare the series of rationals with the series of points on a straight line, we see that (as was known to the ancient Greeks) there are points which correspond to incommensurable lengths, to which no rational number corresponds; in other words, a straight line is richer in points than the series of rational numbers in numbers.

At this point, the temptation becomes strong to say that a convergent series with no rational limit must define a finite *ir*rational number as limit; and, until not very long ago, it was the almost universal practice to define a real number as 'the limit of a convergent sequence of rational numbers.' Yet to do this involves one in a simple logical error, which was first avoided by Weierstrass and Méray, then, in a form substantially identical with Méray's, by Cantor, and, in another form, by Dedekind.§

This logical error is that a number is defined as the limit of a sequence, while the proof that the sequence *has a limit* at all implies that a real number bearing a certain relation to the series has been defined. We cannot, until we have introduced the real numbers, give any valid reason why a convergent series should always have a limit; we can only prove that, *if* it has a limit, it has only one.||

2.

The essential point is the same in all the theories ¶ of irrational numbers. In Cantor's theory, we start from a collection, a_1,

* See Peano in, *e.g.*, the *Formulario de mathematica* of 1905, pp. 74, 83, 95, 100; Ch. Méray, *Leçons nouvelles sur l'analyse infinitésimale*, 1re partie, Paris, 1894, pp. 3-10; Russell, *The Principles of Mathematics*, vol. i., Cambridge, 1903, pp. 149-150, 229, 374, 376-380; and Couturat, *Les principes des mathématiques*, Paris, 1905, pp. 79-81, 138.

† *Not* 'expressions (or signs) for lengths'; see below, § 5.

‡ A sequence such that, given any positive rational ϵ, there is an integer n such that $|s_n - s_{n+m}| < \epsilon$, for any m. That convergency is *necessary* for the existence of a limit is easily proved; that it is also *sufficient*, Bolzano and others have tried to prove (see Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 153, pp. 41-43, 107), but requires a prior arithmetical definition of the 'real numbers.'

§ For references, see *Encycl. des sci. math.*, i. 3, pp. 147-155.

|| We notice that here, if $m > n$, then $s_m \geqq s_n$; but not if the condition of convergency is satisfied only for *some* (but an infinity of) m's, then the sequence has many limits (Cauchy, P. du Bois-Reymond, Hadamard).

¶ We except the theory of Weierstrass, which has the advantages mentioned below. Méray's theory forms an apparent exception, if the term 'irrational number' is thought, in this theory, to have no meaning in itself, but only the phrase 'the variant v has a certain relation with a fictitious number V' to have the meaning 'there is no rational

$a_2, \ldots a_\nu, \ldots$, of some of the rational numbers, which fulfils a certain condition; in Dedekind's theory, we start from a section (*Schnitt*) in the totality of rational numbers; but, in both cases, we are to *create* with our minds a new individual to be 'defined by' our series or our section, and call it a 'real number.'* The advantage of this method over the older would-be arithmetical ones † is that our new 'real number' is not created by 'the summation of an infinite number of terms' (a limit-process which must be defined by real numbers as already-present entities), or as the result,—the existence of which is not proved,—of any process of going to the limit; but we create a new object with a definite position, in respect of magnitude, among the rational numbers—or rather the entities which arise when we substitute for each rational number (r) the fundamental-sequence (the class) $r, r, \ldots, r, \ldots$.‡

We cannot, then, define real numbers as limits, unless the conception of limit is defined in the manner of Peano.§ If,—the term 'limit' being still undefined,||—a denotes a class of rational numbers,¶ Peano introduced a new entity, denoted at first by Ta,** and afterwards by $l'a$,†† defined by three definitions of its magnitude-relations with the rational numbers. If x denotes a rational number:

$x < l'a$ means 'there is a member of a greater than x';

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number to which v has this relation.' Then we evidently cannot speak of 'the fictitious numbers' as if they were, by this, defined entities. But Méray's theory seems (like Cantor's) to consist in the (arbitrary, though convenient and non-contradictory) postulation of new entities ("nombres fictifs") (cf. *Encyclopédie des sci. math.*, i. 3, pp. 148, 149, 152 note 56).

* Cantor said that we "coordinate to the fundamental-series (a_ν) a number b to be defined by it" and defined the magnitude-relations between two such b's by relations between their corresponding series; b can then be proved to be the limit of (a_ν) (*Math. Ann.*, Bd. xxi., 1883, p. 567); see also Dedekind, *Stetigkeit und irrationale Zahlen*, p. 14.

† Namely, in which a real number is defined as the 'limit of a certain series of rationals.' On *geometrical* ones, see below, § 5.

‡ For Méray's statement of this, see *Encyclopédie des sci. math.*, i. 3, p. 149; for Cantor's and Heine's, see Heine's paper in the *Journ. für Math.*, Bd. 74, 1872. For the criticism relating to this, see Russell, *op. cit.*, pp. 270, 282, 285.

§ "Arithmetices Principia nova methodo exposita," Turin, 1889, pp. 15-16, in the various editions of the *Formulaire des mathématiques* (e.g., *Formulario de mathematica*, 1905, p. 105), and the article "Sui numeri irrazionali," *Riv. di mat.*, t. 6, pp. 126-140. Peano's logical symbolism, in which these are written, has been also described in Whitehead's paper in the *Amer. Journ. of Math.*, vol. xxiv., 1902, pp. 367-394. As to the questions of the logical validity of this definition ("by abstraction") and the ("nominal") definition of real number, due to Weierstrass (see the text), Frege and Russell (*op. cit.*, pp. 270-286; see especially the remarks on Peano on pp. 274-275: cf. Couturat, *op. cit.*, pp. 36-43).

|| Hence, in Peano's system, we cannot introduce the idea of rational numbers being limits before studying limits in general, while the other methods have the (didactic, principally) advantage of allowing this. It is, then, not an error from Peano's point of view to speak of "irrationals as based on limits" (Russell, *op. cit.*, p. 274) though it is from (*e.g.*) Cantor's point of view.

¶ The magnitude-relations of these numbers are supposed to have been defined.

** '*Terminus summus*' or '*limes summus classis a*' (*Arith. Princ.*, p. 15).

†† 'Upper limit of *a*' (cf. *Formulario*, 1905, p. 105).

$x = l'a$ means 'there is no member of a greater than x, and, if u is any rational number less than x, there is a member of a greater than u';

$x > l'a$ means 'neither $x < l'a$ nor $x = l'a$.'

The 'real numbers' are then defined to be all such 'upper limits' (l').

From the Cantor and Dedekind point of view of this creation of numbers,* which appears at first (see below) to be logically irreproachable, the unprovable nature of the corresponding Cantor-Dedekind axiom† appears evident, since we cannot create new ‡ points in space—if space is a 'reality.' But further, it is not necessary to 'create' even new numbers; they are there already, in the form in which Weierstrass defined them, and later mathematicians § have emphasised, as *the classes* of the rational numbers which are here fundamental.

Cantor said that b is "defined by" (a_ν), but did not say how. Heine satisfied this need by saying that b is a mere *sign* for (a_ν); ‖ but this laid him open to the charge that can rightly be brought against the formalists, that they mistake the visible *sign* for an essential characteristic of the concept, and also to du Bois-Reymond's ¶ charge that, by this, "analysis would be degraded to a mere game with signs." **

Weierstrass had avoided this formalism by considering his real numbers, not as *signs for* certain groups of rational numbers, but as these groups themselves; and this view, which avoids both the consideration of numbers as signs for geometrical lengths, and signs divorced from their signification, has been

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* Russell seems to me to give a wrong impression of Dedekind's theory when he stated (*op. cit.*, p. 280) that this theory "is designed to prove the arithmetical existence of irrationals." It was designed to *create* or postulate irrationals in a definite way, and then to prove the existence of *limits*.

† Namely, that to every limit of a convergent sequence belongs a line of that length.

‡ For the points in question *may* be absent; intuition is not so refined as to be able to decide on the point.

§ Especially Russell

‖ According to Pringsheim (*Encykl. der math. Wiss.*, i. A3, p. 54, note 21), Cantor (*Math. Ann.*, Bd. 21, p. 553) had a different opinion. The passage referred to was directed against the tendency of some (like Kronecker) who regarded all extensions of the number-concept as "marks of calculation" (*Rechenmarken*). It seems that Cantor did, at this time, support formalism (cf. Cantor in *Math. Ann.*, Bd. xxi., 1883, pp. 589-590, Heine's paper and Cantor's remarks ["Zur Lehre vom Transfiniten," Halle, 1890. pp. 20-21, 54], and Frege's ["Die Grundlagen der Arithmetik," Breslau, 1884, p. 108] note on the character of the analogous [see Cantor, *ibid.*, pp. 34-35, 48-49] transfinite ordinal numbers), but abandoned it,—at least for whole numbers—afterwards (cf. Cantor's criticism of Helmholtz and Kronecker, *ibid.*, pp. 15-20).

¶ *Die allgemeine Functionentheorie.* Tübingen, 1882, p. 55.

** Pringsheim supported the view that the "real *numbers* are an unlimited system of *signs*, which have a uniquely determined *succession*, and with which we can *calculate* according to definite rules" (p. 79 of his essay, "Über den Zahl- und Grenzbegriff im Unterricht," *Jahresber. der deutsch. Math.-Ver.*, Bd. vi., 1898, pp. 73-83; see also *Encykl. der math. Wiss.*, i. A3, pp. 54-55. H. Hankel was the best-known supporter of this formalism (see his *Theorie der complexen Zahlensysteme*, Leipzig, 1867).

adopted in some text-books,* and shows itself, on closer logical consideration, to be what we require for a definition of real numbers.†

3.

Cantor's definition, whose equivalence with Dedekind's is easily proved, is somewhat simpler for our purpose of defining here the real numbers. Modified so as to avoid the dangerous confusion of *numbers* with *signs*, referred to above, this definition runs:

If we have a series $a_1, a_2, \ldots, a_\nu, \ldots$ of rational numbers such that, given a positive rational ϵ, as small as wished, an integer n can be found such that $|a_{n+m}-a_n| < \epsilon$, whatever the integer m may be, we say that this class $(a_1, a_2, \ldots, a_\nu, \ldots)$ *is* ‡ a real number which we may denote b. If, now, to make analogues of the rational numbers a_ν in the class of numbers such as b defined as classes of rationals, we define, as corresponding to the rational number (a *relation*) a_ν, a real number $b_\nu = (a_\nu, a_\nu, \ldots, a_\nu, \ldots)$, which evidently satisfies the above condition; we can prove that $|b-b'_n|$§ diminishes to zero‖ for n great enough, and hence that b can rightly be called the limit of the series $a_1, a_2, \ldots, a_\nu, \ldots$.

4.

I shall now deal with the discussions in this *Gazette*. I learn from Prof. Elliott's last note¶ that his object was to prove this sufficiency *on Dedekind's foundations*. This can, of course, be done, and Dedekind's own proof is, I think, shorter, simpler, and more fundamental than Prof. Elliott's. But in Prof. Elliott's first note,** there was no indication that Dedekind's, or indeed any other, theory of irrationals was adopted. I, supposing that it was an attempt to prove the sufficiency of the criterion without any such theory, pointed out†† that such a 'proof' must be invalid. I happened to use Cantor's theory, and so Mr. Picken ‡‡ accuses me of implying that "a certain order of ideas [presum-

* Cf. Dini and Lüroth, *Grundlagen für eine Theorie* ..., Leipzig, 1892, pp. 2, 6; and M. Godefroy, *Théorie élémentaire des séries*, Paris, 1903, p. 1, said: "... all the other numbers can be defined as groups of integers." This is not quite correct, as (*e.g.*) rationals are *relations*.

† This important question of the existence of real numbers has been emphasised and solved by both Frege (*op. cit.*, pp. 114-115) and Russell (*op. cit.*, pp. 270-286), and consists in that, when real numbers are defined as *classes*, each such class can be shown to have at least one member.

‡ Not 'defines,' with Cantor, without the necessary explanation of *how* it defines it, nor 'is a sign for' with Heine (cf. § 2).

§ $b-b'_n$ is defined to be the class $(a_1-a_n, a_2-a_n, \ldots)$.

‖ The *real* number zero is the class $(0, 0, \ldots, 0)$.

¶ *Gazette*, July, 1906, vol. iii., pp. 349-350.

** *Ibid.*, October, 1905, pp. 296-297.

†† *Ibid.*, May, 1906, p. 327.

‡‡ *Ibid.*, July, 1906, pp. 333-335.

ably Cantor's] is obligatory." I assumed the well-known fact of the essential equivalence of all valid theories of irrationals, and consequently only implied that a logically correct order is obligatory. I do not think I ought to be blamed for pleading that the horse should be put before the cart.

5.

The importance of the theory of irrational numbers and of the proof of the existence of a limit cannot be sufficiently emphasised. The logical error is one which, in Cantor's words,* "has been, I believe, generally overlooked in earlier times because it is one of the rare cases in which actual errors lead to no more important ones in calculation. Nevertheless, I am convinced that all the difficulties which have been found in the concept of the irrational depend on this error, and, by avoiding it, the irrational number is established in our mind with the same definiteness and clearness as the rational number." It may be added that there is an exact analogy between the creation of an irrational number and the creation of the first transfinite ordinal number which Cantor has denoted by ω.†

Thus, in a logically correct treatment of mathematics in its analytical (or, better, logical or arithmetical; in which logic, which suffices for defining the number-concept, *alone* is fundamental) aspect, a theory of irrational number is an indispensable preliminary; and, if logical correctness is not only our ideal in teaching, but also if we carefully avoid, in teaching, giving accounts of things which the logical development of mathematics shows to be false, there is no escape from these somewhat abstract discussions; it will probably be found best to emphasise, after De Morgan's example,‡ the difference between the system of rational numbers and the system of points on a straight line, to introduce real numbers as "Schnitte" (or, more exactly, as classes of all the rationals which satisfy certain conditions),§ and then, for purposes of calculation with sequences $u_1, u_2, \ldots, u_\nu, \ldots,$ of finite real numbers, to deduce the criterion for the existence of a limit of such a sequence,—a criterion which follows immediately from Cantor's theory. But the latter is, perhaps, less to be recommended for teaching purposes.

Another alternative is to regard number-*signs* (*not* numbers) as merely signs for lengths on a straight line, which is supposed to be given in intuition. It is not to be denied that this view,

* *Math. Ann.*, Bd. xxi., 1883, p. 566. † See a note in § 2 above.

‡ *The Connexion of Number and Magnitude; an Attempt to explain the Fifth Book of Euclid*, London, 1836.

§ Thus, $\sqrt{2}$ is the class of those rationals x such as $x^2 < 2$. Cantor's $\sqrt{2}$ is a class of *certain* of these x's, and, though, in Cantor's definition, there is a certain amount of arbitrariness in the choice of elements of the class, yet two such classes are defined to be 'equivalent,' and one name ($\sqrt{2}$) is given o them both.

which was that of P. du Bois-Reymond, has much to recommend it; it is easily grasped, and appears natural, since it was undoubtedly the consideration of linear magnitudes that provided the motive for the introduction of fractions and irrational numbers, and can be used to give significance to negative numbers. But there are three decisive reasons against the validity of this geometrical view: in the first place, the 'continuity' of the line considered can, and should, be defined in logical (arithmetical) terms;* in the second place, an unnecessary indefinable (other than the notions of logic) is included in the foundations,—this is the concept of 'linear magnitude'; and, in the third place, it is possible to define transfinite cardinal numbers which are greater than the cardinal number of the points on any line,—and there would be no place for such numbers in du Bois-Reymond's scheme.

If, then, we decide for a purely arithmetical introduction of irrationals, there is still one more error to guard against,—the tendency to regard numbers as 'signs.'

6.

It is a curious fact that some, even eminent, mathematicians,† when they have desired to emphasise their thesis that number is quite independent of any spatial or temporary intuition, have seen no alternative but to say that numbers are mere signs.‡ Now, we may study signs *quâ* signs *for* something, or we may feel an interest in signs *quâ* signs, and study the ink it is printed in, the material it is printed on, and so on. But these mathematicians carefully avoid committing themselves to the statement that their 'numbers' are signs *for* anything, and, of course, it goes without saying that no 'number' in mathematics can be seriously maintained to be affected by the paper on which it may be written.

And again, a 'variable' in mathematics is, according to Stolz and Pringsheim, a 'sign.' The fact is that we have got into the habit, which tends to shortness and often to usefulness, of talking of x being a '(real) variable' from thinking of a point varying in position along a certain straight line; but the purely arithmetical meaning of 'x is a real variable' is: 'let u denote the class of (all or certain) real numbers, then, if x be *any* member of u, then ...'; and the consequence may be, for example, the proposition: '"x fulfils a certain condition" is not false for all values of x.' The geometrical notion of variability is used when

* See *Encycl. des sci. math.*, i. 3, pp. 146-147, 157-158.

† Cf. § 2 above. We may add the following references to Pringsheim's support of the "sign"-theory: *Sitzungsber. der math.-phys. Cl. der Kgl. bayer. Akad. zu München*, Bd. xxvi., 1896, p. 606, and Bd. xxvii., 1897, pp. 321-324.

‡ The origin of this may be that we say habitually '2 is a number' when we should say the *sign* for a number; but the first is usually understood.

an implication is stated which involves such purely logical notions as *some*, *any*, or *every*.*

To return to numbers. When we say 'x is a number,' x is a sign, but a sign for a number, not a sign for a sign. To take the *sign* as fundamental is the same thing as taking the ordinal *words* (first, second, ...) as fundamental in a theory of numbers, whereas they are, of course, the most unessential part—a way actually followed by Helmholtz and Kronecker, and protested against by Cantor.† A number may be regarded, as Cantor and probably most mathematicians still do, as a product of our mental activity; or we may avoid, as it is desirable to avoid, all psychology by defining it logically as a class, as Weierstrass, Lüroth, and, with greater clearness and consciousness of the issue, Frege and Russell; or again, we may consider numbers to be geometrical entities, like du Bois-Reymond. The third is demonstrably too narrow; the first has disadvantages (in the greater number of indefinables—for we make use of the indefinable 'mind'—required to found mathematics) as compared with the second; while the second requires, I think, some alteration to make it quite free from contradiction;‡ but any of these views is incomparably superior to that view based on a confusion of signs with the things signified, and making, if believed in, of analysis a trivial letter-game which is not even amusing.

7.

In teaching, it seems to me that here also we have evidence which points to the historical method being the only really satisfactory one. For, if we consider the point at which Weierstrass had arrived in his formulation of arithmetical concepts, the doctrine that numbers are 'signs' appears to be a backward step.§ This curious tendency should, of course, be noticed in a historical course, for it serves the useful end of a warning, brings out more clearly the excellency of Weierstrass's conceptions, and shows that one who really grasped these conceptions could hardly have been so unhistorical as to have fallen back into the ancient, because really of date 1778, as Cantor has remarked, idea that numbers are *names*.

8.

One other result of modern investigation into the meaning of 'number' must be mentioned. The separation of analysis from

* Cf. G. Frege, "Was ist eine Funktion?," *Boltzmann—Festschrift*, Leipzig, 1904, pp. 656-666.

† *Zur Lehre vom Transfiniten*, Halle, 1890, pp. 15-20.

‡ I mean that the notion of *class* has limits of validity.

§ De Morgan, for instance, had avoided this error in his text-books.

geometry dates, most explicitly, from Lagrange (1797),* but the justification of this step could only be really † given in quite modern times. The possibility and advantage from the point of view of *method* of separating analysis from geometry were emphasised by Bolzano in 1817,‡ and probably on these grounds the purely arithmetical development of analysis, of which a brilliant and characteristic example is the second edition (1893) of Jordan's *Cours d'analyse*, was raised; but this arithmetical tendency was first really justified by the explicit recognition, due, in its fulness, to Frege and Russell, that the whole of mathematics (including even the so-called 'geometries') follows from purely *logical* indefinables, and that, consequently, to make use of intuition, for example, is only to increase unnecessarily the prerequisites of mathematics.§ This is the true meaning of those somewhat vague words: "good method."

PHILIP E. B. JOURDAIN.

MATHEMATICAL NOTES.

249. [I. 2. b.]

Mr. C. C. Wiles asks (No. 67, p. 167) about numbers N, such that $1/N$, $1/N^2$, etc., have the same number of figures in scale of radix r.

In a paper "On the Period-Length of Circulates" (in the *Messenger of Mathematics*, Vol. XXIX., 1900, pp. 145-179), the present writer has given a discussion of the subject (Art. 10), and has given a list of twenty-five cases, such that l is the *least exponent* giving

$$r^l \equiv +1 \text{ (mod } N^{t-1} \text{ and } N^t\text{), with } r < N^{t-1},$$

the same problem as asked for. Twenty-seven cases are there printed, but two are erroneous, and *should be cancelled*, viz.,

$$44^{13} \equiv +1 \text{ (mod } 53 \text{ and } 53^2\text{)}; \quad 60^{35} \equiv +1 \text{ (mod } 71 \text{ and } 71^2\text{)}.$$

To the above twenty-five may now be added six more, making up thirty-one in all.

$$\left.\begin{array}{l} 43^{102} \equiv +1 \text{ (mod } 103 \text{ and } 103^2\text{)};\\ 100^{243} \equiv +1,\ 175^{162} \equiv +1,\ 307^{243} \equiv +1 \text{ (mod } 487 \text{ and } 487^2\text{)}; \end{array}\right\} \text{due to Mr. Th. Gosset.}$$

$$252^{12} \equiv +1 \text{ (mod } 997 \text{ and } 997^2\text{)}; \quad 390112^4 \equiv +1 \text{ (mod } 17^5 \text{ and } 17^6\text{)}.$$

It will be seen that the limitation $r < N^{t-1}$ is everywhere imposed. This limitation is not observed in Mr. Wiles' examples. Without this limitation it is easy to multiply examples from the tables on pp. 161-179 of the paper above quoted: in fact—

If $r^l \equiv +1$ (mod N^t), then $r^l \equiv +1$ (mod N^τ with $\tau < t$), and in very numerous such cases l will be the Haupt-exponent of r (mod N^τ) if $r > N^\tau$. No case has been discovered of $2^l \equiv +1$ (mod N^{t-1} and N^t), with l as minimum; nor of $r^l \equiv +1$ (mod N^{t-2}, N^{t-1}, and N^t), with $r < N^{t-2}$; but without this limitation it is easy to find them; *e.g.* $57^4 \equiv +1$ (mod 5, 5^2, 5^3).

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* *Théorie des fonctions analytiques*, ..., Paris, 1797.

† Lagrange's motive was probably the economy of thought resulting from the substitution of analytical for geometrical processes.

‡ See Ostwald's *Klassiker der exakten Wissenschaften*, Nr. 153, pp. 4-7. 39.

§ Couturat's book cited above is a clearly written exposition of Russell's work.

When the modulus is a power of 2, then, generally, if $r=2^q-1$,

$$r^2=(2^q-1)^2\equiv+1 \pmod{2^q \text{ and } 2^{q+1}}; \text{ and } r<2^q.$$

This appears to be the only case for which a general rule is known (under the above limitation). ALLAN CUNNINGHAM, Lt.-Col. R.E.

251. [L$_1$. 19. d.] *Some properties of the Conic treated by other methods.*

1. If we represent the distance from a fixed point by r_1 and that from another fixed point by r_2, the equation of the central conic may be written in the form $r_1\pm r_2=2a$, where the two fixed points are the foci of the conic. Differentiating, we get $\frac{dr_1}{ds}\pm\frac{dr_2}{ds}=0$, but $\frac{dr_1}{ds}$ and $-\frac{dr_2}{ds}$ are the cosines of the angles which the tangent makes with the focal distances respectively, and, therefore, the tangent bisects the angle between the focal distances, it being the external bisector in the case of the ellipse and the internal bisector in the case of the hyperbola, and we have, therefore, two confocals intersecting at right angles.

2. Suppose we have two confocals $r_1+r_2=2a$ and $r_1-r_2=2b$. At points of intersection $r_1=a+b$ and $r_2=a-b$, which represent two circles with the foci as centres. If $2c$ be the distance between the foci, the condition that the two circles should intersect in real points is that $a>c>b$, which is, therefore, also the condition that the two confocals should intersect in real points. If we suppose the two points $r_1=a+b$ and $r_2=a-b$ as fixed, we have only two confocals, viz. $r_1+r_2=2a$ and $r_1-r_2=a-b$ passing through them.

3. The following propositions on two confocals passing through P are easy deductions from the principle indicated (S, S' as usual representing the two foci).

(i) Rectangle $SP.S'P$ = difference of squares on the semi-axes of the confocals.

(ii) $SP^2-S'P^2$ = rectangle contained by the axes of the two confocals.

(iii) $SP^2+S'P^2$ = twice the sum of the squares on the semi-axes of the two confocals.

Joining (i) to a known property, we have the square on the semi-conjugate diameter = difference of squares on the semi-axes of the confocals.

Also from (iii) we can show that the sum of the squares on the semi-axes of the confocals $=CP^2+CS^2$, where C is the centre.

Hence as CS is constant, the locus of P is a circle if the sum of the squares of the axes of the confocals through P is given.

4. The equation of a conic may also be written in the form $r=ex$, where r denotes the distance from a fixed point and x the distance from a fixed straight line.

The fixed point is the focus, and the fixed straight line is the directrix. When $e=1$ the equation represents a parabola; for the ellipse $e<1$; and for the hyperbola $e>1$. We may if we choose let r represent the distance from a fixed circle, then if a denote the radius of the circle, writing the equation in the form $r+a=e\left(x+\frac{a}{e}\right)$, we see that the equation represents a conic, the focus being at the centre of the circle and the directrix parallel to the line from which the distances are measured, and at a distance $\frac{a}{e}$.

5. Differentiating, we have $\frac{dr}{ds}=e\frac{dx}{ds}$, and therefore the angle which the focal distance makes with the tangent is $>$ $=$ or $<$, the angle which the tangent makes with the axis according as $e<=>1$. If for ds we write dt, we get a similar proposition about velocities. Also if G be the point where the normal at P meets the axis, $\sin SPG=e\sin SGP$ or $\frac{SG}{SP}=e$.

6. Let P, P' be two consecutive points on the conic, then

$$GG' - e^2dx = e^2PP' \sin SGP.$$

Also
$$\frac{PP'}{\rho} = \frac{GG' \sin SGP}{\rho - PG};$$

$$\therefore \frac{1}{\rho} = \frac{e^2\sin^2 SGP}{\rho - PG},$$

$$\text{or } \rho = \frac{PG}{1 - e^2\sin^2 SGP} = \frac{PG}{1 - e^2\sin^2 PSG} \cdot \frac{SP^2}{PG^2}$$

$$= \frac{PG^3}{PG^2 - e^2SP^2\sin^2 PSG}$$

$$= \frac{PG^3}{SP^2 + SG^2 - 2SP \,.\, SG\cos PSG - e^2\sin^2 PSG \,.\, SP^2}$$

$$= \frac{PG^3}{SP^2(1 + e^2 - 2e\cos PSG - e^2\sin^2 PSG)}$$

$$= \frac{PG^3}{SP^2(1 - e\cos PSG)^2} = \frac{PG^3}{l^2},$$

where l is the semi latus rectum. P. N. DUTT.

252. [L_1. 17. e.] *A Case of Double Contact.*

The equation of any conic having double contact with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the chord of contact being $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \lambda\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right)^2$; when λ is so chosen that the terms of the highest degree form a perfect square and the point $x'y'$ is on the ellipse, the above represents the parabola of closest contact at $x'y'$. The condition for this is that

$$\left(\frac{1}{a^2} - \frac{\lambda x'^2}{a^4}\right)\left(\frac{1}{b^2} - \frac{\lambda y'^2}{b^4}\right) = \frac{\lambda^2 x'^2 y'^2}{a^4 b^4},$$

$$\text{or } \frac{1}{a^2b^2} = \frac{\lambda}{a^2b^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right); \quad \therefore \lambda = 1.$$

Extending the above result to the case of the conic given by the general equation, we see that the equation of the parabola of closest contact is given by $S\Delta + CT^2 = 0$ (Mathematical Tripos, Cambridge, 1906), where S is the equation of the conic, Δ the discriminant, C its first minor with respect to C, and T is the equation of the tangent at a point. For, proceeding as in the previous example, we have

$$\{a - \lambda(ax' + hy' + g)^2\}\{b - \lambda(hx' + by' + f)^2\}$$
$$= \{h - \lambda(ax' + hy' + g)(hx' + by' + f)\}^2,$$

$$\text{or } ab - h^2 = \lambda\{a(hx' + by' + f)^2 + b(ax' + hy' + g)^2$$
$$- 2h(ax' + hy' + g)(hx' + by' + f)\}$$
$$= \lambda\{[ab - h^2][ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy']$$
$$+ af^2 + bg^2 - 2fgh\}$$
$$= \lambda\{(ab - h^2)(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c)$$
$$- c(ab - h^2) + af^2 + bg^2 - 2fgh\}$$
$$= -\lambda\Delta; \quad \therefore \lambda = -\frac{C}{\Delta}.$$

P. N. DUTT.

QUERIES.

(35) Wanted : a proof that every "rule and compass" construction can be performed by the compass alone. S. EDGE.

(36) Who first used the expression "golden section"? AUR. SECT.

(37) Wanted : a list of French and other foreign periodicals dealing with elementary mathematics. JOURNAUX.

(38) Can we by elementary methods construct a triangle ABC, given the base BC, the median from A, and $B \sim C$? Q.

(39) Given two tangents to a conic and their points of contact, with a normal to the conic through the intersection of the tangents; describe the conic geometrically. C. N.

(40) How can the mathematical use of the word *involution* be explained etymologically? I.

(41) Find by geometry the vertex A of a triangle ABC, given in magnitude and position the base BC and the intersection with AB of the symmedian from C. SIM.

(42) Where shall I find a discussion of the relative propriety of the terms "binomial" and "binominal"? B. I.

(43) Who first used the term "moment" of a force? A. R. C. H.

(44) Who discovered the "parallelogram of forces"? P. O. F.

(45) Given the images of A, B, C respectively in the opposite sides of the triangle ABC, construct the original triangle. Has this problem been solved? References will be welcomed. JAY.

(46) Does any algebraical relation exist between e and π. P. E.

(47) A strip of paper or ribbon is tied in a flat knot (drawn tight up). Prove that it will assume the form of a regular pentagon.

A *direct* solution is invited. [It is easy to prove that the regular pentagon is a *possible* form, and by the way the knot is tied the form is unique, $\therefore$ etc.] G. H. B.

(48) (*a*) The length of a line "read off from a scale correct to the hundredth of an inch is 12·25 inches." Should the limit of error be taken as ·005 or ·01? Possible reasons for the latter value are that if the scale is graduated correct only to hundredths of an inch the graduation may be ·005″ in error, and the error in reading may also be ·005″ making a total error of ·01″. Yet solutions published to this question (which was set in last Cape Matriculation) assume ·005 as the limit of error.

(*b*) We are asked to write the result of a problem "*correct to as many decimal places* as possible, and to indicate the limits of error in the answer so written." The result is $\cdot 465 \pm \cdot 0005$, and therefore lies between ·4645 and ·4655. Writing these correct to three places, we get ·465 and ·466, to two places, ·46 and ·47, to one place ·5 and ·5. Must the answer be stated as ·5 (with $-\cdot 0355$ and $-\cdot 0345$ as limits of error) since ·46 does not represent the higher value correct to two places, nor ·47 the lower value?

(*c*) A similar point occurs in this. The following are correct to the third place. To what place can their sum be relied on, and what are its limits of error? 1·414, 4·143, 1·732, 7·312, 3·415, 100·888, 1·999, 9·191. IOTA.

ANSWERS TO QUERIES.

[7, p. 95.] This question can be reduced to the intersection of a circle and a cubic which passes through the centre of the circle. This solution is to be found in the *Nouvelles Annales*, 1855, p. 413. ED.

[9, p. 95.] It may be worth noting that

$$[2(ab+cd)(ad+bc)-(ac+bd+ef)(a^2+b^2+c^2+d^2-e^2-f^2)]$$
$$\times[ab+cd-ef]\equiv[(ab+cd)e-(ad+bc)f]^2.$$

The right-hand member is positive; $ab+cd-ef$ is positive since the quadrilateral is convex. Hence the inequality stated. ED.

[10, p. 95, and Note, p. 132.]

Prof. Cayley's addition to the note in the *Messenger* was not an 'explanation,' but a means of testing whether factors of the so-called envelope were extraneous or not. The example made to illustrate the note was so obvious (as Mr. W. D. Evans points out) that no test was needed. As more attention is now deservedly paid to the logic of our fundamental theorems, I venture to repeat the argument of the note. All that can be inferred from the combination of $\phi(x, y, a)=0$ and $\frac{\partial\phi}{\partial a}=0$ is *some locus* passing through the intersections of $\phi=0$ and the next member of the family. If then $\phi=0$ pass through fixed points, the envelope, as usually obtained, will contain factors corresponding to *loci going through* the fixed points, and having otherwise nothing to do with the geometrical envelope. R. W. GENESE.

[22, p. 131.] M. Poussin has published a book entitled *Sur l'Application des procédés graphiques aux calculs d'assurances* (Dulac, 8, rue Lamartine). An article by L. Lalanne *De l'emploi de la Géométrie pour résoudre certaines questions de moyennes et de probabilités* appeared in the *Journal de Mathématiques pures et appliquées*, III. Series, Vol. V., 1879, pp. 107, 123. Poudra and Hossard published an in-8° in 1819 entitled: *Question de Probabilité résolue par la Géométrie.* A few geometrical solutions are to be found, if we remember rightly, in Czuber's *Probabilités et Moyennes geométriques* (Hermann, Paris). ED.

[23, p. 131.] In *Knowledge*, Vol. XIV., p. 210, appear the formulae

$$\underline{|n}=(\cdot 37n+1)^n \quad \text{where} \quad 13<n<650,$$
$$\underline{|n}=(\cdot 368n+1)^n \quad \text{where} \quad 650<n<11,000.$$

For $n=50$, we get $\underline{|n}=316\ldots$, (65 digits)

whereas Stirling's formula gives

$$\underline{|n}=3041409\ldots.$$

Prof. Forsyth in the 53*rd Brit. Assoc. Report*, 1884, pp. 407-8 gives

$$\underline{|n}=\sqrt{2\pi}\left[\frac{\sqrt{n^2+n+\frac{1}{6}}}{e}\right]^{n+\frac{1}{2}}. \quad \text{The error is } \frac{1}{240n^3},$$

$$\text{i.e.} < \frac{1}{20n^2} \text{ of the error in using Stirling's formula.}$$

A. Pellet in the *Comptes Rendus* (1903, May 4) shews that $\log\underline{|n}$ lies between $\sqrt{2\pi}\log(x/e)^x-1/(24x)$ and $\sqrt{2\pi}\log(x/e)^x-1/(24x)+7/(2^6.5.9.x^3)$, where $x=n+\frac{1}{2}$.

Also $\log\sqrt{2\pi}[xe^{-1}(1+24^{-1}x^{-2})^{-1}]^x$ lies between these limits and gives $\underline{|2n}$ with an error less than $1/(600x^3)$. ED.

[23, p. 131.] When n is very large, $n!$ is approximately equal to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, a result known as Stirling's Theorem. See Chrystal's *Algebra*, Part II., pp. 368, 590, etc.

A comparatively simple way of obtaining the result is by operating on

$\log n$ with the two sides of the equation for the operator $\frac{d}{dn}$, obtained by putting $\frac{d}{dn}$ for x in Bernouilli's expansion, thus:

$$\frac{x}{e^x-1}=1-\frac{1}{2}x+\frac{B_1}{2!}x^2+\dots.$$

Hence $$\frac{\frac{d}{dn}}{e^{\frac{d}{dn}}-1} \text{ is the same as } 1-\frac{1}{2}\frac{d}{dn}+\frac{B_1}{2!}\frac{d^2}{dn^2}+\dots;$$

$$\therefore\ \frac{1}{\Delta} \text{ is the same as } \frac{1}{\frac{d}{dn}}-\frac{1}{2}+\frac{B_1}{2!}\frac{d}{dn}+\dots;$$

$$\therefore\ \frac{1}{\Delta}\log n=\int\log n\,dn-\frac{1}{2}\log n+\frac{B_1}{2!}\frac{d}{dn}\log n+\dots;$$

$$\therefore\ \sum_1^{n-1}\log n=\int\log n\,dn-\frac{1}{2}\log n+\frac{B_1}{2!}\frac{1}{n}+\dots;$$

$$\therefore\ \log(n-1)!=n\log n-n-\frac{1}{2}\log n+\text{const., when } n \text{ is large};$$

$$\therefore\ \log(n-1)!=\log\frac{n^n}{e^n}\frac{1}{\sqrt{n}}C, \text{ when } n \text{ is large};$$

$$\therefore\ n!=C\sqrt{n}\left(\frac{n}{e}\right)^n, \text{ when } n \text{ is large.}$$

C can be found from Wallis's expression for π. JAMES STRACHAN.

[**24**, **p. 165**.] *Mantisa* or *Mantissa*, according to Lewis and Short, is a Latin (Tuscan) word signifying "a worthless addition, a makeweight." The decimal part of the logarithm is evidently the excess or surplus. ED.

[**25**. **p. 165**.] According to Chasles (*Aperçu Historique*, p. 494), an Arabian, Albategnius, prince of Syria "eut l'heureuse idée de substituer aux *cordes* des arcs . . . les demicordes des arcs doubles, c.à.d. les *sinus* des arcs proposés." A suggestion is made in André's *Nouveau Cours de Trigonométrie*, that the half chords were written *semisses inscriptae*, whence the contraction *s-ins*, and finally *sinus*. This seems as ingenious as a derivation of *cadaver*—ca = *caro*, da = *data*, ver = *vermibus*. ED.

[**27**, **p. 166**.] The following is attributed to Lagrange by Serret (*Cours d'Algèbre Supérieure*, 5th Edition, p. 41, § 20).

If $x=\frac{E+\sqrt{A}}{D}$, then, starting from the complete quotient, $x_n=\frac{E_n+\sqrt{A}}{D_n}$, ($E_n$ and $D_n>0$), it is shewn that $D_n<2\sqrt{A}$, and $a_n<2\sqrt{A}$ where a_n is the incomplete quotient corresponding to x_n.

Thus E_n, D_n, a_n are limited, and the complete quotients x_n can have only a *finite* number of different values. Hence after a number of operations, not greater than $2\sqrt{A}\times\sqrt{A}$ or $2A$, we reach a complete quotient already obtained; after this the rest of the series of complete or incomplete quotients will be formed of the same sequence or *period* of terms already found, and this goes on indefinitely. ED

[**28**, **p. 166**.] If the vertices D, E, F of the equilateral triangles are described all outwards or all inwards, bisect the sides of the triangle DEF at L, M, N, and describe equilateral triangles on the sides of LMN, all inwards if the original equilateral triangles were outwards, and all outwards if the original ones were inwards. The vertices of these last triangles will coincide with A, B, C, the vertices of the required triangle.

Another method is: Describe equilateral triangles on *DEF* itself (inwards if *D*, *E*, *F* were outwards, and *vice versâ*, as before). A triangle is thus obtained whose mid-points are the required points *A*, *B*, *C*. A. LODGE.

[**29, p. 166.**] The solution for concurrent bisectors depends upon an equation of the 14th degree; for non-concurrent bisectors this equation is of the 16th degree. In the general case they are irreducible. For the similar problem, given the feet of the symmedians, the equation is of the 12th degree. ED.

REVIEWS.

La Formule $\rho = re^{\iota(\phi+\iota\psi)}$ interprétée géométriquement dans l'espace, de manière à prendre la forme d'un Quaternion. Par J. H. PEEK, Docteur ès Sciences, conseiller actuariel de la Banque Royale d'Assurance à Amsterdam. (H. Eisendrath, Amsterdam, 1907.)

The author starts from the formula $e^{\iota\phi} = \cos\phi + \iota\sin\phi$. By putting $\phi = \frac{\pi}{2}$, we obtain $e^{\iota\frac{\pi}{2}} = \iota$; and by raising this to the power ι, $e^{\iota\iota\frac{\pi}{2}} = \iota^\iota$. The author decides that, while this formula is valid, it is not permissible to multiply the ι's in the index $\iota\iota\frac{\pi}{2}$; so that it does not yield the result $e^{-\frac{\pi}{2}} = \iota^\iota$. He gives as a reason that he regards ι^ι as being of a vectorial nature. He also regards ι as a vector, and as having a direction perpendicular to the ϕ-plane, contrary to the usual practice of choosing it to be in the ϕ-plane. This gives rise to many questions, such as: If ι is a vector in $e^{\iota\phi}$, what meaning can be attached to $e^{\iota\phi}$? and if ι is not a vector in $e^{\iota\phi}$, how are we to distinguish when ι is a vector and when it is not? The author does not pause over such questions, and gives no answer to them.

The argument proceeds as follows: From the formula $e^{\iota\frac{\pi}{2}} = \iota$, we have $e^{\iota\phi} = \iota^{\frac{2\phi}{\pi}}$, and $e^{\iota\iota\phi} = (\iota^\iota)^{\frac{2\phi}{\pi}}$. This must surely rejoice the hearts of all index jugglers. But even they may be a little startled at the next step: with reference to the ϕ-plane, $e^{\iota\phi} = \cos\phi + \iota\sin\phi$; hence with reference to the ψ-plane, $e^{\iota\iota\psi} = (\iota^\iota)^{\frac{2\psi}{\pi}} = \cos\psi + \iota^\iota\sin\psi$. Moreover, by defining $\phi + \iota\psi$ to be the angle made by two radii drawn at inclinations ϕ and ψ to the line of intersection of the ϕ-plane and ψ-plane, the result $\cos(\phi+\iota\psi) = \cos\phi\cos\psi$ is obtained. The object of the thesis is to represent $e^{\iota(\phi+\iota\psi)}$ as a so-called exponential quaternion obeying the laws of ordinary algebra, and serving as a bridge between ordinary algebra and the calculus of quaternions proper. But until the author has thrown some light on the points referred to above, his thesis cannot be taken seriously. F. S. MACAULAY.

Advanced Examples in Physics. By A. O. ALLEN, B.Sc.

Examples in the Mathematical Theory of Electricity and Magnetism. By J. G. LEATHEM, D.Sc. (London: E. Arnold.) Each, 1s. 6d.

Mr. Allen's collection contains examples in all branches of Physics, chiefly from London and Victoria University examinations. These are nearly all numerical, and answers are given. Several of the problems in general physics should interest teachers of mechanics.

Most of Dr. Leathem's 205 examples are taken from Tripos and College examinations at Cambridge. They are of very varying difficulty, ranging from simple numerical applications of Ohm's law up to really difficult conundrums.

C. S. J.

(*a*) **Geometrical Conics.** By F. S. MACAULAY. Second edition. Pp. x, 300. (Cambridge: University Press. 1906.)

(*b*) **The Elements of the Geometry of the Conic; with a chapter on the geometry of certain curves occurring in applied mathematics.** By

G. H. BRYAN and R. H. PINKERTON. Pp. xii, 270. (London: Dent & Co. 1907.)

Apollonius has not been canonized, so that it is possible to write a book on geometrical conics without any fear of committing sacrilege, or deranging a consecrated order of propositions. These two books illustrate very well the variety as well as the considerable amount of agreement in the views of those who are actually teaching the subject. The work of Messrs. Bryan and Pinkerton is the more 'practical' in a way, and keeps closer to the algebraical theory (compare the treatment of diameters): both books deal pretty fully with orthogonal projection, and include curvature. It may be added that in each case as much success has been achieved as can be reasonably expected in making the proofs independent—a point that cannot be wholly neglected from an examination point of view.

The special feature of (b) is the supplementary chapter (pp. 224-70), which contains brief but useful notes on the catenary, cycloid, equiangular spiral, etc. Most of the exercises are wholly or partly of a practical character. The authors say, in this connexion, that they believe that the geometry of the conic "lends itself admirably—possibly even better than the substance of Euclid—to the illustration, by means of constructive exercises and problems, of fundamental principles and of the relations which are deduced from them."

Dr. Macaulay's work appeals, in part, to a more advanced type of student: it includes the discussion of the actual sections of a right circular cone, a good deal about confocals (including Graves's theorem), and three excellent chapters on poles and polars, involution, and homographic correspondence respectively. The treatment is partly algebraical in essence: thus the general definition of a cross-ratio is given, and the main theorem of correspondence is proved from the equation $(ABCX)=(A'B'C'X')$. I suppose it is to be inferred from this that a boy learns the facts more easily and quickly in this way than if the method of Reye were followed and only harmonic relations defined in the first place. It should be specially noticed that Dr. Macaulay has adopted von Staudt's definition of a conic as the locus of points in a plane polar system which lie on their polars. This is undoubtedly the most scientific definition of a conic from the standpoint of modern geometry, and it is very interesting to find it in an English school book. Theorem 14 (p. 276), which is due to von Staudt, and of great importance, is very properly included. On the whole, this last chapter, which has been entirely rewritten, strikes me as an admirable introduction to von Staudt's beautiful theory, and well worth the trouble which it must have taken to compose. Even the algebraic part, referred to above, will be helpful to those who find the purely geometrical method difficult to grasp.

Many examples have been given, with notes on the harder ones. There is a beauty about the better sort of example in geometrical conics which, to me at any rate, surpasses anything else in the whole range of mathematics; and it is very refreshing to find that exercises of this kind are not yet wholly banished in favour of graphs and calculus, useful as the latter are in their proper place.

Both (a) and (b) seem satisfactory in matters of printing, etc., though (a) perhaps has the advantage, especially in the lettering of the diagrams. The figures in (b), with their very big and bold letters, irresistibly remind one of the blackboard, where legibility at a distance is so essential that letters of an exaggerated size have to be used and the points to which they refer indicated, when necessary, with a stick. But in a book it is quite possible (as in Dr. Macaulay's book) to combine legibility with proper location.

In conclusion, I confess, in a very contrite spirit, that it is entirely my fault that no notice of Dr. Macaulay's book has appeared in the *Gazette* before. I undertook to review the first edition, and forgot all about it.

G. B. MATHEWS.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

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